MATH 496 - The Beauty of Cubics

David An

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1 Introduction

In this paper, we study the nature of the zeroes of polynomials and their representation inside the complex plane. We observe that the plotting of the real and imaginary parts of the roots of a polynomial inside the complex plane yield many vibrant patterns.

2 Background

2.1 Introduction to Polynomials

A polynomial is an expression which involves as a sum of powers in one or more coefficients multiplied by coefficients. The degree of the polynomial is defined as the highest degree in the terms of the polynomial. The number of variables in the polynomials is defined as the number of unique variables which are not coefficients inside the polynomial.

The polynomials limited to this study are polynomials in one variable of degree n. In addition to that, the coefficients of the polynomial are randomly generated sets of integers of equal bounds. For example, given a bound of b and degree n, we see that a polynomial $f(x)$ would be of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x^1 + a_0 \quad \{a_1, a_2, ..., a_n\} \in [-b, b]$$

The definition of the polynomial above is the definition we will use throughout the remainder of the paper.

2.2 Properties of Polynomials

Polynomials have many interesting properties that make them worth studying. We take the definition of the polynomial above and define some properties of polynomials.

2.2.1 Fundamental Theorem of Algebra

For any polynomial that is in our set of polynomials, we see that there are at most n different solutions to a polynomial of n degrees. This can be done in two approaches. The first proof that a degree n polynomial has at most n solutions can be attributed to the Fundamental Theorem of Algebra. It states that a polynomial can be factored into n linear factors (multiplicity considered) of

$$f(x) = (x - r_1)^{j_1}(x - r_2)^{j_2}...(x - r_n)^{j_n}$$

where r_i is the root and j_i is the multiplicity of the root. However, we can prove this also without using the Fundamental Theorem of Algebra. Before going into that, we state a consequence of the Fundamental Theorem of Algebra which is that a polynomial is always guaranteed to have at least 1 root.

Theorem 2.1. *A polynomial has exactly n roots, multiplicity withholding.*

Proof. To prove that a polynomial has exactly n roots, multiplicity withholding, we first observe that the case for a constant function is trivial and a polynomial of first degree ($f(x) = a_0 + a_1 x$) has a root of $x = -\frac{a_0}{a_1}$. Now, we proceed by induction. Assume that we have a polynomial $f_{n-1}(x)$ of degree $n - 1$ has $n - 1$ roots.Let us have a polynomial now $f_n(x)$ of degree n. By a result of the Fundamental Theorem of the Algebra, we have that the polynomial f_n has at least one root, r. Thus, we can write $f_n(x) = (x - r)f_{n-1}(x)$. Since f_{n-1} has $n - 1$ roots, we see that $f_n(x)$ has n roots. $\square$

2.2.2 Guaranteed real zeroes of odd degree polynomials

Another unique property of polynomials begins to introduce properties in the complex numbers. Since we are graphing the polynomials in the complex plane, each polynomial root has both a complex and real part. Depending on the polynomial, are we able to see that there are real roots guaranteed? For our case, the polynomials p are defined over the field $p \in \mathbb{Z}[x]$. One point of the argument is that conjugates of root hold. Because the coefficients of the polynomials are real numbers ($\mathbb{Z} \subset \mathbb{R}$), we are able to use the following theorem:

Theorem 2.2. *The complex-conjugate root theorem allows us to state that if the coefficients of the polynomial are real and $p(a + bi) = 0$ for $a, b \in \mathbb{R}$, then $p(a - bi) = 0$ as well.*

For the case of odd degree polynomials, we show down below:

Theorem 2.3. *Any odd degree polynomial in $\mathbb{Z}[x]$ has a guaranteed real zero.*

Proof. Let us take a polynomial of degree d where d is an odd number (odd degree polynomials). Let us also define a complex root $r = x + yi$ where $r \in \mathbb{C}, x, y \in \mathbb{R}$. Now, let $f(r) = 0$, where r is a root. For a polynomial of degree greater than 1, say a 3^{rd} degree polynomial $f_3(x) = ax^3 + bx^2 + cx + d$ where plugging in the root, we get that $f_3(r) = a(x+yi)^3 + b(x+yi)^2 + c(x+yi) + d = a((x^3 - 3xy^2) + (3x^2y - y^3)i) + b((x^2 - y^2) + 2xyi) + c(x + yi) + d = 0$. Since we know that the quantity evaluates to 0, we see that the imaginary parts all become zero and we are left with the real parts. Now, let us take that same with $r' = x - yi$. We see that $f_3(r') = a((x^3 - 3xy^2) - (3x^2y - y^3)i) + b((x^2 - y^2) - 2xyi) + c(x - yi) + d = 0$. Notice that only the signs of the imaginary components change. Thus, whenever r is a root, r' is a root as well. Thus in any polynomial, we have $2n$ imaginary roots at most. Because we have an odd degree polynomial and by 2.2.1, we see that a real root is guaranteed for any polynomial of odd degree. □

Now, what if the polynomial of an even degree? The question of that remains much more complicated than odd degree polynomials. Analytically, we can see that for a real root to occur, there needs to be an interval $X = [a, b]$ and a subsequent interval $Y = [b, c]$ where the signs differ (i.e. all values of X lie above the real axis and all values of Y lie below the axis). Now, suppose we have such an even polynomial $f_n(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ with all the coefficients in $\mathbb{Z}$. One can see that the end behavior of the even polynomial depends on the coefficient of the leading term. If the leading term of $f(x)$ is $a_n > 0$, then, $\lim_{n\to\pm\infty} f(x) > 0$. Similarly, if the leading term of $f(x)$ is $a_n < 0$, then, $\lim_{n\to\pm\infty} f(x) < 0$.

In addition to that, we see that the constant term in a polynomial a_0 depends on the shift of the polynomial vertically. Thus, we can form the following hypothesis: given an even polynomial of n where $f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$, there exist at least 2 real roots when $a_na_0 < 0$. From a high level, one can see that $a_na_0 < 0$ only when they have opposing signs. In $a_n > 0, a_0 < 0$, we see the graph fill be shifted below the real axis however still trend up, thus crossing the real axis and implying the existence of real roots. The same is said for the other case when $a_n < 0, a_n > 0$. Now, we attempt a proof as follows:

Theorem 2.4. *Given any polynomial $p \in \mathbb{Z}[x]$ of an even degree, then there exists at least 2 real roots when the product of the leading coefficient and the constant term is less than zero, $a_na_0 < 0$.*

Proof. Let us define a continuous polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ where $n = 2j$. When $a_n > 0, a_0 < 0$, we see that at $f(0) = a_0$. We know that if $a_n > 0$, then, $\lim_{n\to\pm\infty} f(x) > 0$. By the Intermediate Value Theorem, we can see that we will have a zero in both directions, thus the existence of at least 2 real roots. The same can be argued for the other case. $\square$

2.2.3 Polynomials generated by constraints

Finally, one more thing we observe is the number of polynomials that are generated given our constraints. This measure is significant in our processes as the number of polynomials influences the total processing time of calculation. Given a polynomial of degree d, we know we have $d + 1$ terms. In addition to that, we see that for bounds $x, y \in \mathbb{Z}$ for coefficients, we have $2b + 1$ possible values for coefficients. That means, we can calculate based on parameters the number of total polynomials which is equal to $(y - x + 1)^{d+1}$

3 Methods of Solving Polynomials

As seen in 2.2.3, there remain an immense amount of polynomials but solving them remains the challenge. With so many, the efficiency of the solver remains a big priority. With millions of polynomials and if one takes around .3s to solve, we still have thousands of hours of processing time to generate results. Down below, we list some methods of solving polynomials.

3.1 Newton-Raphson's Method

One method to solve for polynomials that are generalized to polynomials of any degrees is using Newton-Raphson's method. Essentially, this method recursively finds and approximates roots of a polynomial. For quadratic and cubic polynomials there are closed form equations (covered later) that make it simpler to solve. However, for quintic and higher degree polynomials, there do not exist friendly equations to solve them. Now, let $f(x)$ be a differentiable and continuous polynomial. Given an interval $[a, b]$, we find an approximation of where a root is, x_0, as a starting point. Next, we solve for the derivative of the polynomial, $f'(x)$. Then, we recursively find roots as follows:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

As $n \to \infty$, the roots will converge to a value.

3.2 Polynomial and Cubic Formula

Next, we visit closed form equations to find the roots of polynomials of 2nd and 3rd degree. Given a second degree polynomial: $f(x) = ax^2 + bx + c = 0$, we know that the roots of the polynomial can be solved with a constant time expression:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Because this is covered extensively in beginner calculus classes, we will not spend as much time on this method.

However, the story for cubics is different. Just adding one power will increase the complexity of the closed form immensely. We will briefly cover the method of solving them. Given a polynomial $f(x) = ax^3 + bx^2 + cx + d = 0$, we can solve the roots with the following equation proposed by Jerome Cardan (1501-1576):

Theorem 3.1. *Given any polynomial in the form of $p(x) = ax^3 + bx^2 + cx + d$, there exists a closed form solution for roots as such:*

$$x = p + \sqrt[3]{p - Q} + \sqrt[3]{p + Q}$$

where

$$
\begin{aligned}
Q &= \sqrt{q^2 + (r - p^2)^3} \\
p &= -\frac{b}{3a} \\
q &= p^3 - \left(\frac{pc + d}{2a}\right) \\
r &= \frac{c}{3a}
\end{aligned}
$$

However, for quadratic and cubic polynomials, it provides for a good solution to solve for something in constant time. However, once we scale up to quintic and larger degree polynomials, the problem becomes much more difficult to solve for. In fact, many larger degree polynomials do not have closed form equations where one is able to plug in coefficients. They only allow for solvability and the existence of a solution, not an exact solution.

3.3 Companion Matrix

Finally, we see the method of using a companion matrix to solve a polynomial. This is a method that generalizes to a polynomial of any degree. This method also remains the most optimal to calculate the roots of a polynomial as matrix multiplication and operations have become very efficient in the past few years on computer systems. Given any polynomial of degree n,, we can construct a $n \times n$ matrix M structured as the following where $a_1, a_2, ...a_n$ correspond with the coefficients of the polynomial.

$$
M_p = \begin{bmatrix}
0 & 0 & \dots & 0 & -c_0 \\
1 & 0 & \dots & 0 & -c_1 \\
0 & 1 & \dots & 0 & -c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 1 & -c_{n-1}
\end{bmatrix}
$$

For example, given the polynomial that $f(x) = x^4 + 3x^3 - 2x^2 + x - 6$, the companion matrix is

$$M = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

We know from Edelman [1] that the eigenvalues of this matrix are the roots of the polynomial. Since matrix multiplication has become extremely efficient in recent years due to advances in computing and this method can be generalized to polynomials of any degree, this remains the best candidate in order to calculate the roots of polynomials in experimentation.

4 Optimization

We cover a (extremely) brief overview of the necessary optimizations performed during this experiment in order to reasonably obtain results. As this is a math paper, we will not be going extremely in depth with optimization techniques.

Because of the exponentially growing number of polynomials to solve for and current computing limitations, we look at different aspects of calculations. Since matrix multiplication has been heavily optimized due to advances in deep learning, and we only do polynomials with a maximal degree of 3, we utilize the `numpy` library to perform all computations. `numpy` is written in Python to compute a variety of tasks which include but are not limited to: matrix multiplication, tensor calculations, and optimization problems.

However, because `numpy` is built on top of Python which has an interpretation layer from Python to C and C to machine code, we automatically run the code in C to skip the translation step. By performing the code at a lower level (C), we are able to speed up computation significantly.

A listing of the code used in the experiment will be available in the appendix.

5 Experiments and Results

In this section, we discuss the outcomes of the study as well as any consequential results that may arise from it. First, let's define the set of polynomials we will be studying. The set that we study will be cubic polynomials with integer coefficients from the set of $C_p = \{-m, -m+1, ..., -1, 0, 1, ..., m-1, m\}$. Which means that we have an upper bound of $3(2m+1)^4$ complex roots due to the Fundamental Theorem of Arithmetic. We generate sets of cubic polynomials iteratively with specifying ranges and plotting the imaginary and real part on the complex plane. After generating a set of cubic polynomials with range of $[-30, 30]$, and plotting the roots in the complex plane with the real axis going horizontally and the imaginary axis going vertically. The following figure is what results:

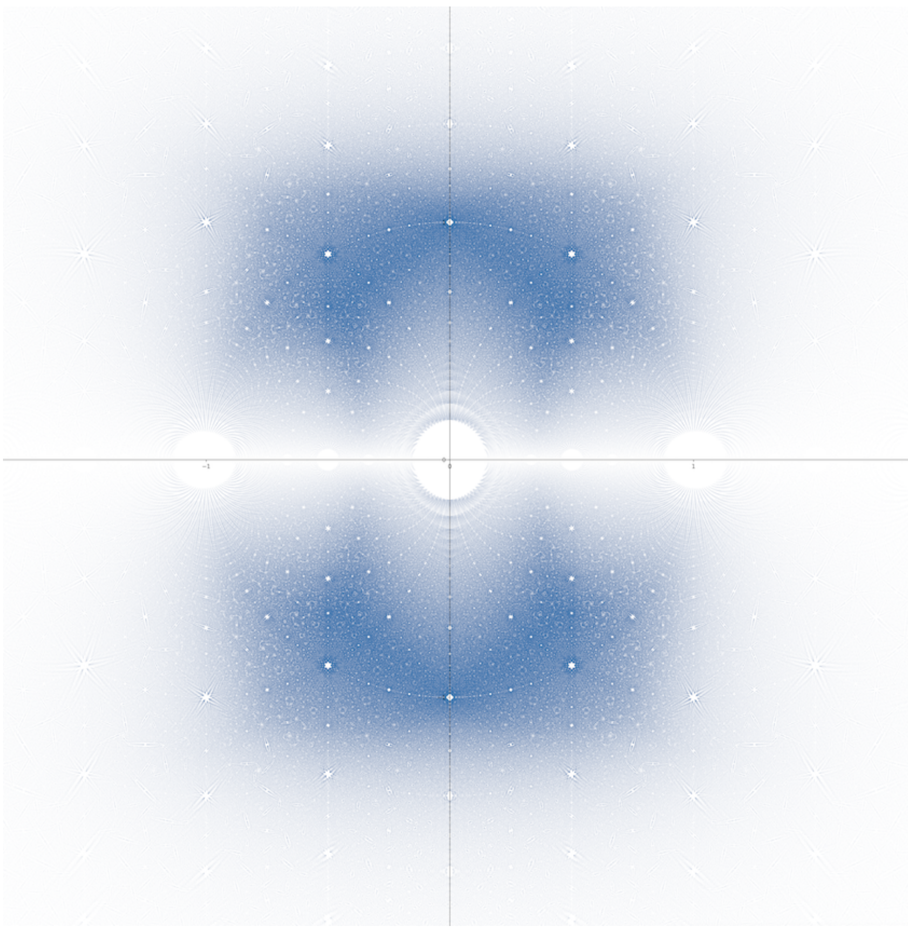


Figure 1: Overall graph of the roots of cubic polynomials

We see that by zooming out, we are able to see a overall pattern which is flowerlike and circular in nature. The reflection across the real axis (horizontal) is due to the reciprocity of the polynomials ($p(x)$ shows that $p(-x) = -p(x)$). However, while overall we see these shaded areas, by zooming into certain points, we are able to see certain patterns.

5.1 The point at $e^{\frac{\pi i}{6}}$

One of the first (and more) notable positions is located at $e^{\frac{i\pi}{6}} = \cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})$. Upon zooming in, we see that there is only a singular root at exactly $\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})$. In addition to that, there seems to be a void where there are no roots within that region. Normally, such a hole would be disregarded, but we see that the hole forms a 6 petaled flower shape that extends in a tesselation-like pattern across a few iterations. Also, we see there seems to be patterns expanding out at each vertex of the petal too.

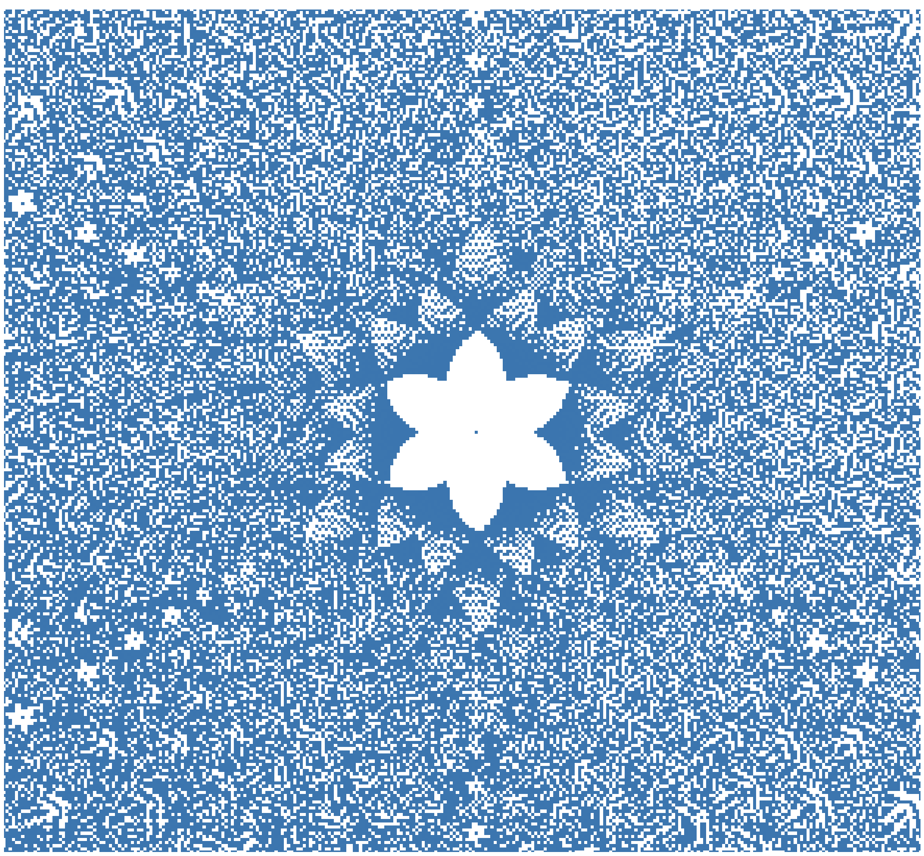


Figure 2: The point at $\frac{i\pi}{6}$

5.2 The point at i

Now, we cover another interesting point at i. This falls just 1 unit vertically on the imaginary axis. Instead of a 6-petaled figure with defined petals, instead we get a 4-petaled figure with rounded edges (semicircle like). The properties are still somewhat analogous to the previous point at $e^{\frac{\pi i}{6}}$. Subsequent patterns extend similarly by tessellating out of the original figure. However, these patterns are not as pronounced as the previous point.

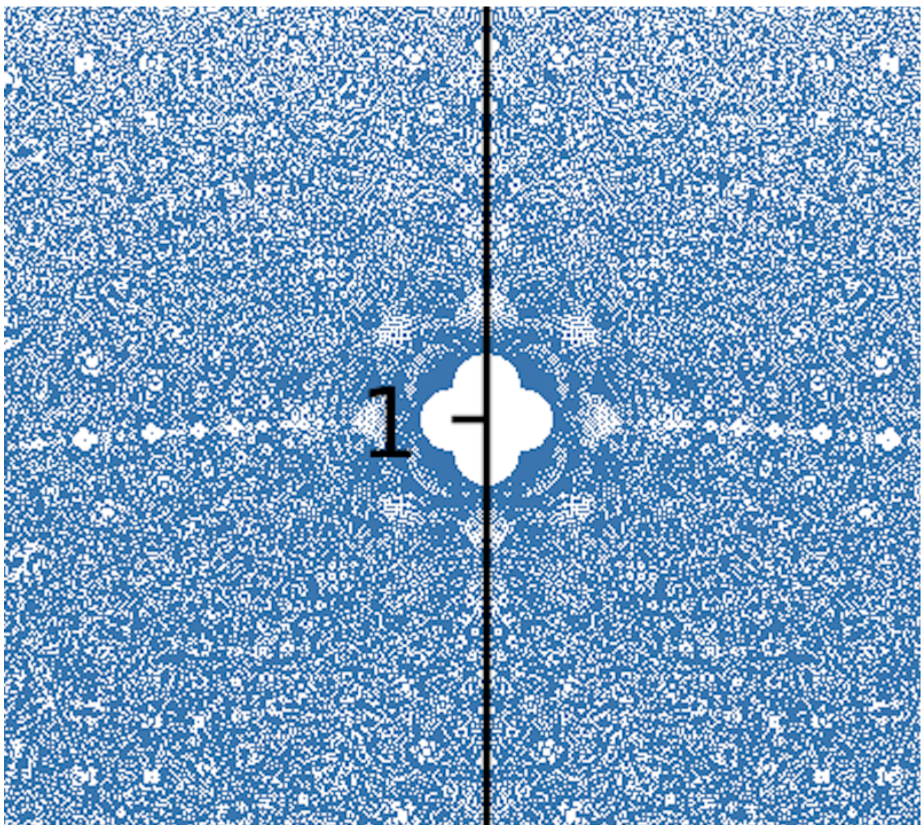


Figure 3: The point at i

5.3 The point at 1

Finally we discuss one more notable pattern that occurs within the graph which occurs at 1 (i.e no imaginary part exists). Here, we deviate from patterns like discussed previously. Instead, at this point, we shift from a polygon based shape

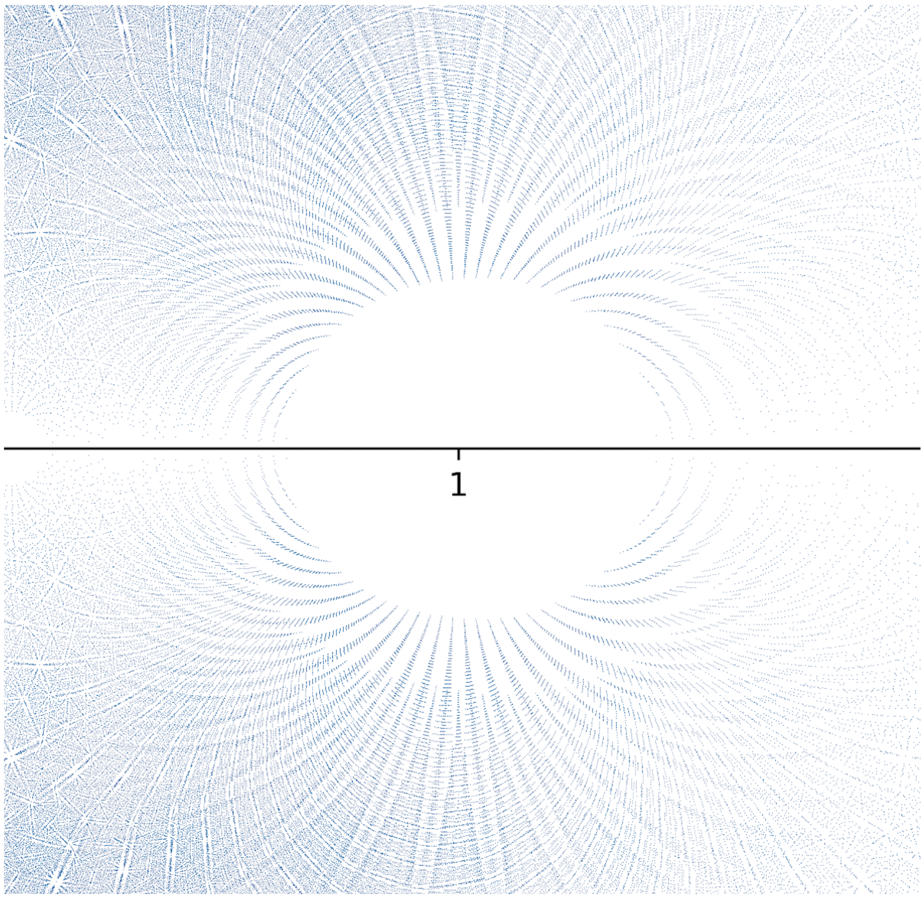


Figure 4: The point at 1

to a more open form shape which resemble feathers.

Besides those 3 points, there remain hundreds of other notable points which we will cover in the appendix. However, those are the most distinct ones upon first glance of the picture.

6 Convergence of Real Parts

With so many roots, is there something we can learn about the distribution of the real and imaginary parts of the root. In this section, we attempt to show that the distribution of the real parts of roots converge to a normal distribution.

6.1 Empirical Evidence

When computing the roots, we store the real and imaginary parts of the roots, along with the actual root itself. We observe that the more polynomials we have (by increasing the bounds), the real parts converge from a bimodal distribution to what appears to be a near unimodal distribution of some form of normal distribution. With testing data in place, we seek to prove something similarly but mathematically.

6.2 Mathematical Evidence

Let us take the set of the real components of roots as $C = \{x_i : i \in I\}$ where I is a index set for the values. We want to construct a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where we can assume a normal distribution. In addition to that, we define the event space as the power set of all the samples. This is due to the power set following the event space axioms. Finally, we make our probability event $\mathbb{P}$ such that it is defined as for any subset of events $A \in \mathcal{A}$, then we have such $\mu(A) = \sum_{x_i \in A} p_i$. At first, this seems like a naive way to do it then by setting the random variable $X : \Omega \to C$ where $X = id$. This is such that we have a function $P[X = x_i] = p_i$. We can see that and $P(X \in A) = P(X^{-1}(A)) = P(A) = \mu(A)$. However, it doesn't seem to give a solid reason why it is not normal. We have only simply created a probability space with a random variable which does not imply normality.

We see that a random variable X is said to have a standard normal distribution if it is absolutely continuous with a density given by:

$$\frac{d\mathbb{P}_x}{d\lambda_1}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Theorem 6.1. *The real parts of polynomial roots follow a normal distribution*

Proof. Let us take the same previous identity as $X = id$ where $X(\omega) = \omega$ in the space. We desire such a probability function such $\mathbb{P}(X \leq x) = \mathbb{P}(\{\omega \in \mathbb{R} : \omega \leq x\})$

$$= \int_{\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

As such, we see that the simplest way to satisfy such a constraint for normality is to define a probability function as above but inside of the desired region this time:

$$\mathbb{P}(x) = \int_D \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

Which we see that this definition satisfies the fact that given this set of points, we can make such a probability space and random variable that has the property of being normal. Along with empirical evidence as we increase the bounds, we make the assumption that the real parts of polynomial roots eventually resemble to a normal distribution. □

This remains as an attempt but does not provide concrete evidence for the case. It is hoped that empirical evidence and such a construction provide sufficient evidence. We do not know if this is definite and guaranteed over polynomials or only in the field and constraints we define as.

7 Bounds of Roots by Coefficients

In this section, we attempt to determine a method of finding a soft bound of roots based on a polynomial. While a majority of the paper has been regarding polynomials over the integers, in this section, we utilize the natural numbers as the field the polynomials are defined over. Given a polynomial $p(x) \in \mathbb{N}[x]$ with degree n, $a_0 \neq 0$ which implies that $a_0 \geq 1$ and finally, all the coefficients bounded by $a_i \leq N$ $a_i \in \{a_1, a_2, ...a_{,n}\}$.

Theorem 7.1. *Given a polynomial* $p(x) \in \mathbb{N}[x]$ *with degree* n*,* $a_0 \neq 0$ *which implies that* $a_0 \geq 1$ *and all of the coefficients bounded by* $a_i \leq N$ $a_i \in \{a_1, a_2, ...a_{,n}\}$*, then a root* x *such that* $f(x) = 0$ *has a lower bound of* $1/(N+1)$

Proof. Let us take a polynomial $p(x) \in \mathbb{N}[x]$ with degree n, $a_0 \neq 0$ with coefficients bounded by $a_i \leq N$ $a_i \in \{a_1, a_2, ...a_{,n}\}$. Let there exist a $x \in \mathbb{R}$ such that $p(x) = 0$. By rearranging, we see that $-a_0 = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x$ by the triangle inequality.

We observe that the right-hand side has a value of at most $|N||x|/(1 - |x|)$ since we are bounding all the coefficients by N, it suffices to say that the sum at most has to be $|N||x|$ and dividing by $1 - |x|$ gives extra bounds. In addition to that, we see that the absolute value of the left-hand side ($|-a_0|$) is at least greater than 1 (also by construction). Thus, we end up with

$$1 \leq |-a_0| \leq -a_0 = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x \leq |N||x| \leq \frac{|N||x|}{1-|x|}$$

Thus, we see that

$$1 \leq |a_0| \leq \frac{|N||x|}{1-|x|}$$

Through this statement, it implies that

$$|x| \geq \frac{1}{N+1}$$

□

8 Discussion

Throughout this study, we explore the nature of cubic polynomials with integer coefficients and the representation of their roots inside the complex plane. We study 3 main things: the graphical representation of roots of cubic polynomials inside the complex plane, the distribution of the real parts of roots, and generating a soft bound on the magnitude of roots.

We observe that there are many flower-like patterns within the graph representation as well, however, the reason for such patterns are unknown.

8.1 Future Direction

We can consider many future directions for continuing this study. One of them is instead of using integer coefficients, we can extend the coefficients to the reals as well as subsets of the reals. By extending the subset of coefficients to be used, we can study the distributions of the roots and see if they deviate from integer coefficients.

Additionally, we can also continue into the investigation of why the zeros of polynomials have exclusion zones that resemble flowers or other shapes as well. All of these notable points actually follow on the line that traces out the unit circle which leads us to another point to investigate which is why there are concentric circles and other lattice-like points present.

Finally, one last direction that relates to the study is tightening bounds to the roots of polynomials given the coefficients. Currently, we have determined a (relatively) wide lower bound for roots. Some next steps would be determining an upper bound or tightening the bounds (or determine that it is already the tightest such bound).

9 Appendix

Here, we list other figures which are not covered in the main report. This section is a tribute to the beautiful and (sometimes unexplainable!) patterns we naturally find in mathematics.

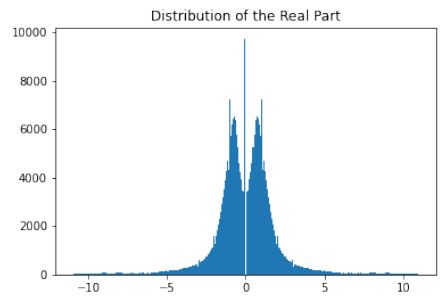
9.1 Code Used in Computation

```
import numpy as np

def generate_coeff(bound, degree):
    candidates = list(range(-bound,bound+1))
    combinations=list(itertools.product(candidates, repeat=
degree+1))
    print('Original Total Polynomials:',len(combinations))

    print('Sampled Total Polynomials:',len(combinations))
    return [list(c) for c in combinations]
```

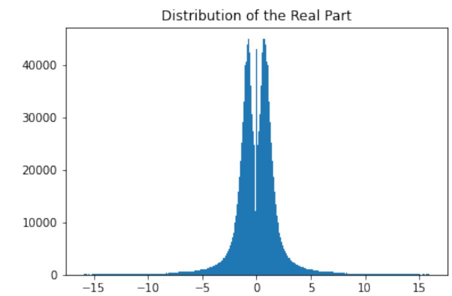
Listing 1: Generating all possible coefficients

```
import numpy as np

def generateRootsNumpy(polynomials):
    print('======= Generating Roots for Polynomials ========')
    returnRoots = []
    start=time.time()
    for p in polynomials:
        added = np.roots(p)
        returnRoots += parseNumpy(added,p)
    end=time.time()
    print("Time Taken:",str(end-start)+"s")
    print("Roots Found:", len(returnRoots))
```

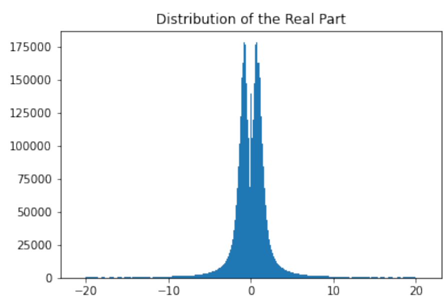
Listing 2: Returning all roots based on a set of polynomials



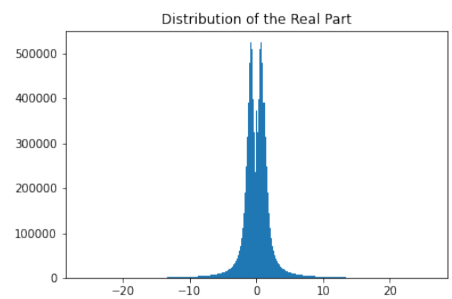
(a) Coefficients bounded to $[-10, 10]$



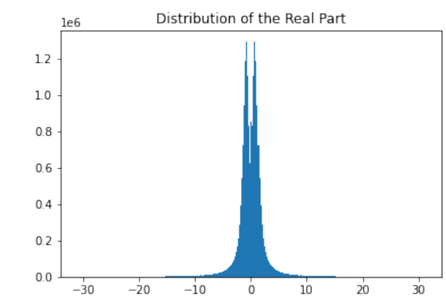
(b) Coefficients bounded to $[-15, 15]$



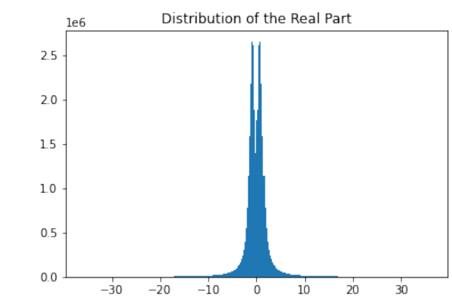
(c) Coefficients bounded to $[-20, 20]$



(d) Coefficients bounded to $[-25, 25]$



(c) Coefficients bounded to $[-30, 30]$



(d) Coefficients bounded to $[-35, 35]$

Figure 5: The distribution of the real parts of polynomial roots in different groups of polynomials

References

[1] Edelman, A., Murakammi, H.: Polynomial Roots from Companioin Matrix Eigenvalues (1994)

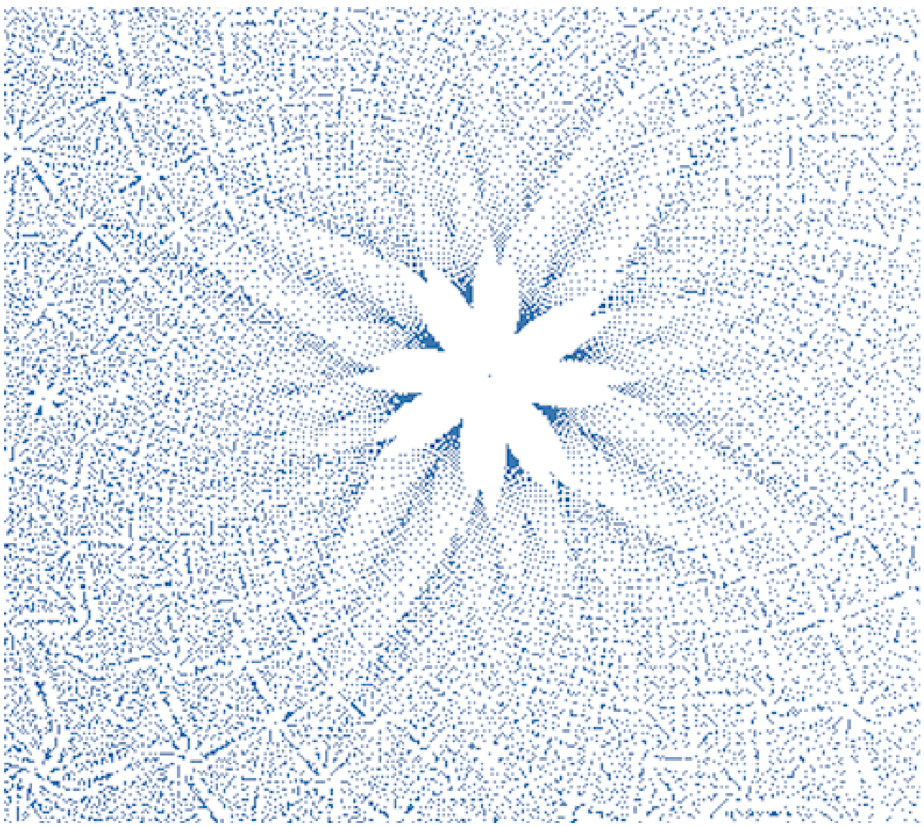


Figure 6: The point at $1+i$. A series of these points fall on a unit circle centered at 1
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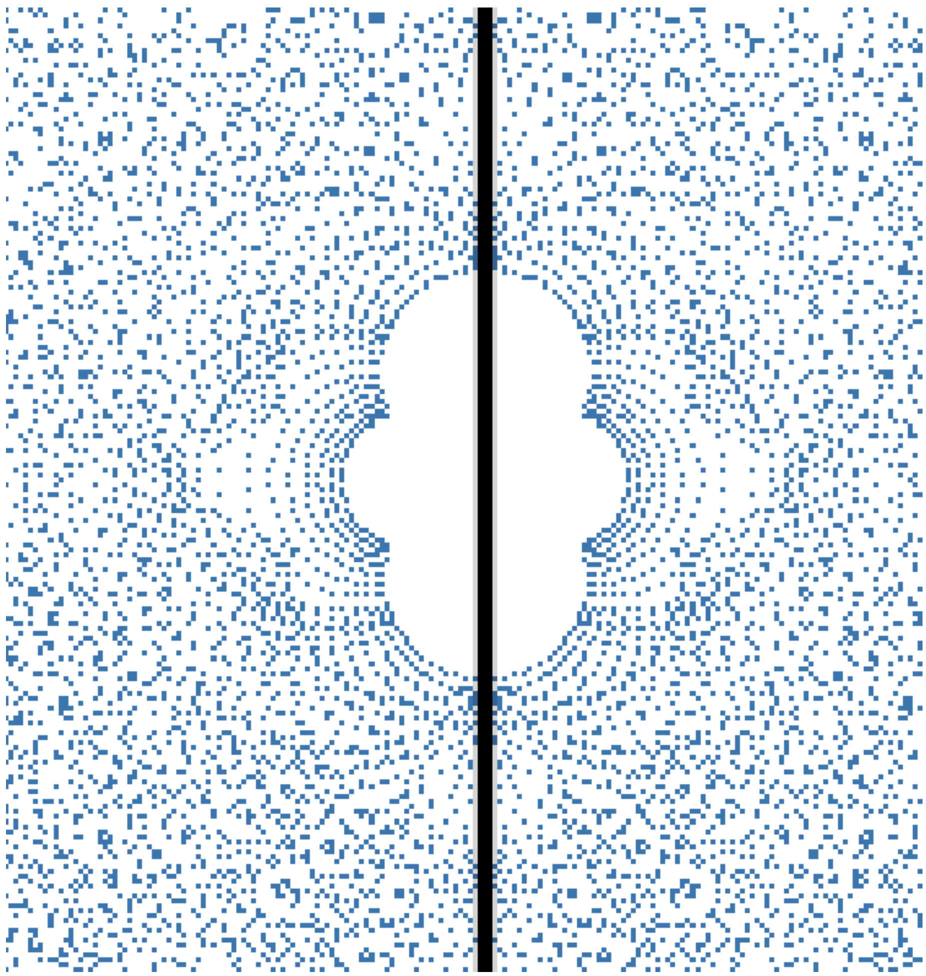

Figure 7: The point above i but an exact determination of location is unknown. This figure remains similar to the found exactly i

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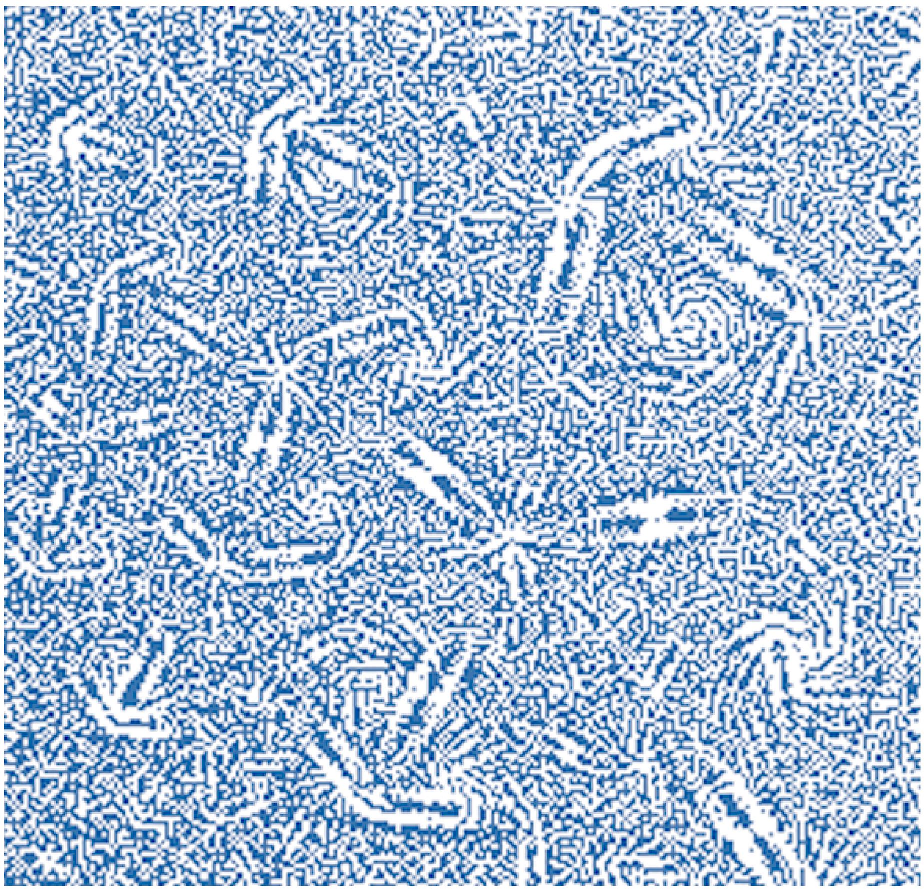

Figure 8: A region with the plane that has numerous interlocking 'Y' shapes that are nested within each other

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